

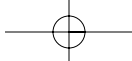
# Modeling Fixed-Income Securities and Interest Rate Options

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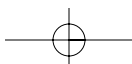
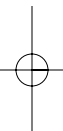
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# Prologue

## APPROACH

This book studies a relatively new approach for understanding and analyzing fixed-income securities and interest rate options that has revolutionized Wall Street. This approach is that of *risk management*—that is, the arbitrage-free pricing and hedging of fixed-income securities. It does not maintain the traditional focus of textbooks in this area. Traditional textbooks concentrate on the institutional setting of these markets. Where the institutional setting consists of a detailed description of the various markets and instruments that trade—for example, market structures, the key players, conventions in quoting prices, and contract specifics—risk management—the pricing and hedging of these securities—is, at best, only an afterthought. Pricing theories are introduced in an ad hoc and sometimes inconsistent fashion. Today, with the advent of sophisticated and readily available computer technology that can be applied to study these securities, the traditional approach is now outdated. Although institutional details are still important, their importance is secondary to risk management concerns.

This book provides a self-contained study of this new approach for pricing and hedging fixed-income securities and interest rate options. This new approach uses modern option-pricing theory and is called the HJM model. The HJM model is used extensively in industry. Heath, Jarrow, and Morton developed the HJM model in a sequence of papers.<sup>1</sup> It was motivated by the earlier

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<sup>1</sup>These papers are given in the references at the end of the chapter. See Jarrow [6] for a review of the history behind the development of the HJM model.

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work of Ho and Lee<sup>2</sup> on this same topic and by the martingale pricing methods of Harrison and Pliska.<sup>3</sup>

The HJM model is presented using the standard binomial model so often used to analyze the pricing and hedging of equity options. The standard binomial model is chosen because of its mathematical simplicity, in comparison to the more complex stochastic calculus techniques. In addition, this choice involves little loss of generality, because the alternative and more complex stochastic calculus techniques still need to be numerically implemented on a computer, and a standard approach for implementing these techniques on the computer is to use the binomial model. So, from an implementation point of view, the binomial model is all one needs to master.

This flexible, new approach to fixed-income securities has revolutionized the industry. The HJM model is already being employed by commercial and investment banks to price and hedge numerous types of fixed-income securities and interest rate options. One reason for this extensive usage is that this model provides a Lego-type building-block technology. With minor modifications, this technology can be easily extended (built on) to handle the pricing of other more complex instruments—for example, foreign currency derivatives, commodity derivatives, and credit derivatives. These extensions will also be illustrated, albeit briefly, in this book.

### MOTIVATION

Examining three figures can motivate the subject matter of this textbook. The first, Figure 1, contains a graph of a yield curve for Treasury securities. For now, do not worry about the exact definition of a Treasury security or its yield. These will be discussed more fully later on in the text. Here, it suffices to understand that Treasury securities are bonds issued by the U.S. government. A bond is an IOU issued for borrowing a stated amount of dollars (for example, \$10,000), called the *principal*, for a fixed period of time (for instance, five years), called the bond's *maturity*. Interest is paid regularly (often semiannually) on this IOU. A Treasury bond's *yield* can be thought of as the interest earned per year from buying and holding the bond until its maturity.

In Figure 1 we see a Treasury yield curve on March 31, 1997. The  $y$ -axis is percentage per year; the axis starts at 4 percent. The  $x$ -axis gives the different bonds' maturities in years. The maturities run from zero to thirty years. As depicted, the yield curve is upward sloping, indicating that it is more costly to

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<sup>2</sup>See Ho and Lee [5].

<sup>3</sup>See Harrison and Pliska [1].

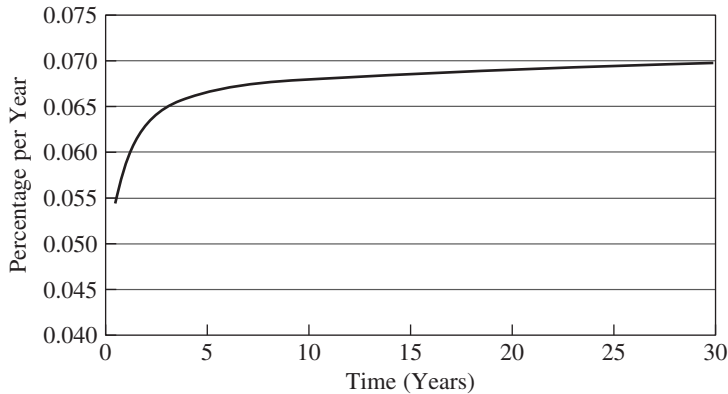


FIGURE 1  
Treasury bond yields on March 31, 1997

borrow for thirty years than for five years than for one year. A higher interest rate is being charged per year, the longer the borrowing horizon. This increasing interest charge reflects the differing risks of the longer borrowing horizons and the market's perception of how short-term interest rates will change in the future.

The Treasury yield curve changes through time. This is illustrated in Figure 2, which repeats Figure 1 but graphs it at different dates over the months January 1992 through January 1997. As seen, the yield curve's shape and height differ at different times over this five-year period. Sometimes the yield curve is nearly flat (see January 1997) or even downward sloping. The shape of the yield curve is referred to as the *term structure of interest rates*. Its fluctuation through time is called the *evolution* of the term structure of interest rates. This evolution is random because it is not predictable in advance. Forecasting the evolution of the term structure of interest rates well is crucial to dependable risk-management procedures. Models for the evolution of the term structure of interest rates form a big part of this book's content.

*Interest rate options* or *derivatives* are financial contracts whose cash flows depend, contractually, on the Treasury yield curve as it evolves through time. For example, one such interest rate derivative is a financial contract that pays its owner on the second anniversary date of the contract, \$10,000 times the difference between the ten-year and five-year yields observed on the two-year anniversary date, but only if this difference is positive. This is a *call option* on an interest rate spread. To understand how to price this spread option, one needs to understand (or model) how the term structure of interest rates evolves through time. The better the model (or forecast) of the term structure evolution, the better the pricing model will be. After all, the expectation of the

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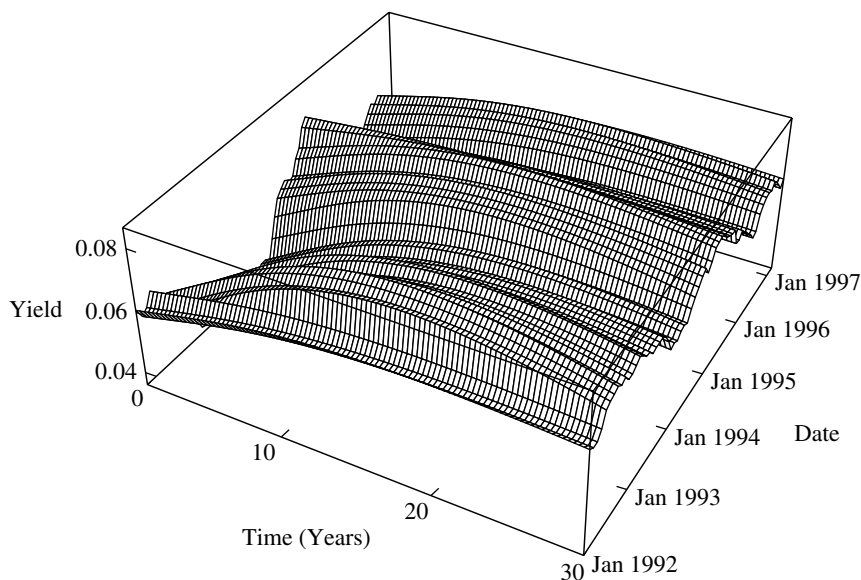


FIGURE 2  
Yields from January 1992 to March 1997

profits or losses from the spread option depends on what the shape of the term structure looks like in two years.

A key theme in this book, therefore, is how to model the evolution of the term structure of interest rates illustrated in Figure 2. Much of our emphasis will be on developing this structure. Then, given this evolution, the second theme is how to price and hedge the interest rate derivatives written against it. This is a nontrivial exercise and reflects the remaining focus of the book. This topic is referred to as *risk management*.

A key characteristic of U.S. Treasury securities is that they are considered to be default free; that is, an IOU issued by the U.S. Government is considered to be safe, with the receipt of the interest and principal payments considered a sure bet. Of course, not all IOUs are so safe. Corporations and government municipalities also borrow by issuing bonds. These corporate and municipal loans can default and have done so in the past. The interest and principal owed on these loans may not be paid in full if the corporation or municipality defaults before the maturity date of the borrowing. This different default risk generates different interest rates for different borrowers, reflecting a credit risk spread, as illustrated in Figure 3.

Figure 3 plots the average yields per year on Treasury, corporate, and municipal bonds over the years 1989 through 1997. The  $y$ -axis is percentage per year, and it starts at 2 percent. The  $x$ -axis is time measured in years from 1989

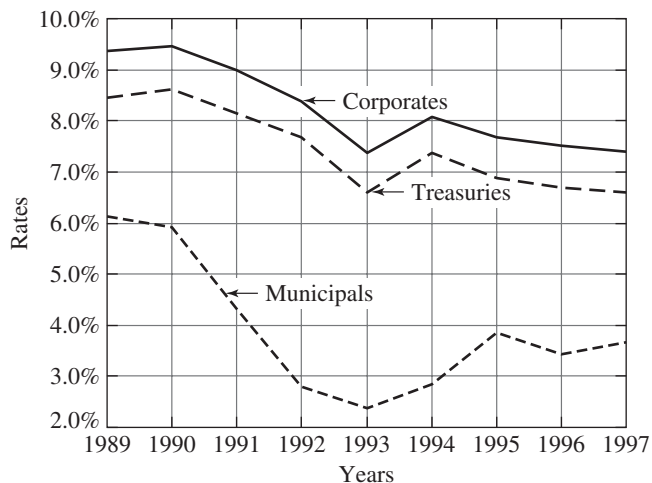


FIGURE 3  
Bond yields

to 1997. As indicated, the cost for borrowing is higher for corporates than it is for Treasuries. Surprisingly, however, the cost for Treasuries is higher than it is for municipals. The corporate–Treasury spread (positive) is due to credit risk. This credit spread represents the additional compensation required in the market for bearing the risk of default. The second Treasury–municipal spread (negative) is due to both default risk and the differential tax treatments on Treasuries versus municipals. Treasuries and corporates are taxed similarly at the federal level. So, tax differences do not influence their spread. In contrast, U.S. Treasury bond income is taxed at the federal level, while municipal bond income is not. This differential tax treatment influences the spread. As indicated, the tax benefit of holding municipals dominates the credit risk involved, making the municipal borrowing rate less than the Treasury rate. Understanding these differing spreads between the various fixed-income securities is an important aspect of fixed-income markets.

Unfortunately, for brevity, this book concentrates almost exclusively on the Treasury curve (default-free borrowing), and it ignores both credit risk and taxation. Nonetheless, this is not a huge omission. It turns out that if one understands how to price interest rate derivatives issued against Treasuries, then extending this technology to derivatives issued against corporates or municipals is (conceptually) straightforward. This is the Lego-type building-block aspect of the HJM technology. Consequently, most of the economics of fixed-income markets can be understood by considering only Treasury markets.

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This situation can be intuitively understood by studying the similarity in the evolutions of the three different yields: Treasuries, corporates, municipals—as depicted in Figure 3. The movements in these three rates are highly correlated, moving almost in parallel fashion. As one rate increases, so do the other two. As one declines, so do the others, although the magnitudes of the changes may differ. A good model for one rate, with some obvious adjustments, therefore, will also be a good model for the others. This fact implies that the same pricing technology should also apply—and it does. Therefore, the same risk-management practices can be used for all three types of fixed-income securities. Mastering risk management for Treasury securities is almost sufficient for mastering the risk-management techniques for the rest. The last chapter in the book will briefly cover this concept.

### METHODOLOGY

As mentioned earlier, we use the standard binomial option-pricing methodology to study the pricing and hedging of fixed-income securities and interest rate options. The binomial approach is easy to understand, and it is widely used in practice. It underlies the famous Black-Scholes-Merton option-pricing model.<sup>4</sup> As shown later, the only difference between the application of these techniques to fixed-income securities and those used for pricing equity options is in the construction of the binomial tree. To understand this difference, let us briefly review the binomial pricing model.

A diagram with nodes and *two* branches emanating from each node on the tree is called a *binomial tree*; see Figure 4. The nodes occur at particular times—for example, times 0, 1, 2, and so forth. The nodes represent the state of the economy at these various times. The first node at time 0 represents the present (today's) state of the economy. The future states possible at times 1 and 2 are random. The branches represent the paths for reaching these nodes, called *histories*.

For risk-management purposes, the relevant state of the economy at each node is characterized by the prices of some set of securities. For equity options, a single security's price is usually provided at each node, the price of the underlying equity. For fixed-income securities, an array (a vector) of bond prices must be considered at each node of the tree, rather than the single equity's price (a scalar) as in the case of equity options. Otherwise, the two trees are identical. Hence, term structure modeling is just the multidimensional extension of the standard equity pricing model. Unfortunately, this multidimensional

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<sup>4</sup>An excellent reference for the Black-Scholes-Merton option-pricing model is Jarrow and Turnbull [7].



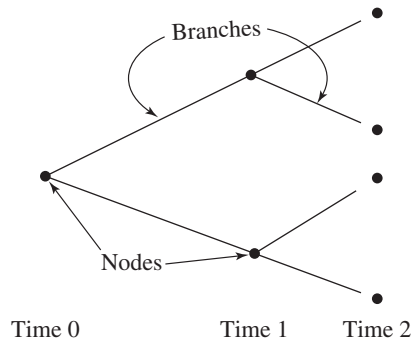


FIGURE 4  
Binomial tree

extension to a vector of prices at each node of the tree is nontrivial, as the complexity comes in insuring that the given evolution of prices is arbitrage free.

By *arbitrage free* we mean that the prices in the tree are such that there are no trading strategies possible using the different bonds that can be undertaken to generate profits with no initial investment and with no risk of a loss. For an equity pricing model, constructing an arbitrage-free binomial tree is trivial. For fixed-income pricing models, however, constructing an arbitrage-free binomial tree is quite complex. Quantifying the conditions under which this construction is possible is, in fact, the key contribution of the HJM model to the literature. Once this arbitrage-free construction is accomplished, the standard binomial pricing arguments apply.

The standard binomial pricing argument determines the arbitrage-free price of a traded derivative security, in a frictionless and competitive market, where there is no counterparty risk (these terms are explained later in the text). It does this by constructing a *synthetic* derivative security, represented by a portfolio of some underlying assets (bonds) and a riskless money market account. This portfolio is often modified, rebalanced, through time in a dynamic fashion. This portfolio's (the synthetic derivative security's) cash flows are constructed to match exactly those of the traded derivative security. But, the synthetic derivative security's cost of construction is known. This cost of construction represents the "fair" or "theoretical" value of the derivative security.

Indeed, if the traded derivative security's price differs from the theoretical value, an arbitrage opportunity is implied. For example, suppose the theoretical price is lower than the traded price. Then, sell the traded derivative (a cash inflow) and construct it synthetically (a cash outflow). The difference in cash flows is positive because the traded price is higher than the cost of

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construction. One pockets this initial dollar difference, and there is no future liability, because the cash flows are equal and opposite from thereon.

The identification of such arbitrage opportunities is one of the most popular uses of these techniques in practice. Other uses are in the area of risk management, and these uses will become more apparent as the pricing and hedging techniques are mastered.

### OVERVIEW

This book is divided into five parts. Part I is an introduction to the material studied in this book. Part II describes the economic theory underlying the HJM model. Part III applies the theory to specific applications, and Part IV studies implementation and estimation issues. Part V considers extensions and other related topics.

Part I, the introduction, comprises two chapters. Chapter 1 gives the institutional description of the fixed-income security and interest rate derivative markets. Included are Treasury securities, repo markets, FRAs, Treasury futures, caps, floors, swaps, swaptions, and other exotics. This brief presentation provides additional references. Chapter 2 describes the classical approach to fixed-income security risk management. The classical approach concentrates on yields, duration, and convexity and is studied for two reasons. First, it is still used today, although less and less as time passes. Second, and perhaps more important, understanding the limitations of the classical approach provides the motivation for the HJM model presented in Part II.

Part II provides the theory underlying the HJM model. Chapter 3 introduces the notation, terminology, and assumptions used in the remainder of the text. The traded securities are discussed, where for the sake of understanding, many of the actually traded financial securities are simplified. Chapter 4 sets up the binomial tree for the entire term structure of zero-coupon bonds. A one-factor model is emphasized in Chapter 4 and throughout the remainder of the text. Economies of two and more factors are also considered. Chapter 5 describes the traditional expectations hypothesis, which, after a curious transformation, becomes essential to the pricing methodology used in subsequent chapters. Chapter 6 introduces the basic tools of analysis: trading strategies, arbitrage opportunities, and complete markets. Chapter 7 illustrates the use of these tools for zero-coupon bond trading strategies with an example. Chapter 8 provides the formal theory underlying Chapter 7. Finally, Chapter 9 discusses option-pricing theory in the context of the term structure of interest rates.

Part III consists of various applications of this pricing and hedging technology. Chapter 10 studies the pricing of coupon bonds. Chapter 11

investigates the pricing of options on bonds (both European and American). Chapter 12 studies the pricing and hedging of forwards, futures, and options on futures. Chapter 13 analyzes swaps, caps, floors, and swaptions. Chapter 14 prices various interest rate exotic options including digitals, range notes, and index-amortizing swaps. At the time of the writing of this book, index-amortizing swaps were considered to be one of the most complex interest rate derivatives traded. Index-amortizing swaps will be seen as just a special, albeit more complex, example for our pricing technology.

Part IV investigates implementation and estimation issues. Chapter 15 shows that the model of Parts I–III can serve as a discrete time approximation to the continuous-time HJM model. Both the continuous-time HJM model and its limit are studied here. Chapter 16 investigates the statistical estimation of the inputs: forward rate curves and volatility functions. Empirical estimation is provided.

Part V concludes the book with extensions and generalizations. Chapter 17 is a discussion of the class of spot rate models, providing an alternative perspective to the HJM approach covered previously. Last, Chapter 18 completes the book with a discussion of various extensions and a list of suggested references. The extensions are to foreign currency derivatives, credit derivatives, and commodity derivatives.

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